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First Passage Time for Stochastic Dynamic System and Climate Modeling

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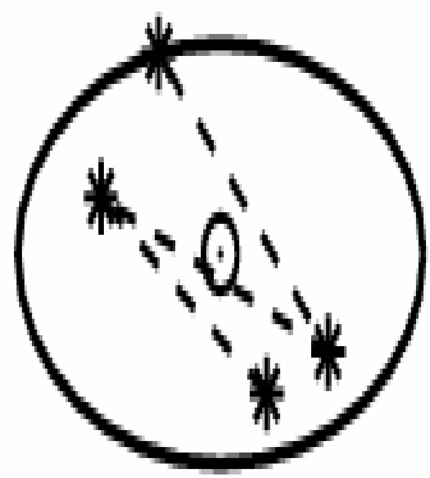
First-Passage Time (FPT)

ρ → Radius

FPT → Time when the particle first passes through the boundary.



FPT in Climate Studies

- (1) Climate Index Prediction
- (2) Climate Model (or Ocean/Atmospheric Model) Predictability

(1) Climate Index Prediction

Spatial Correlation of Surface Pressure of Djakarta (Berlage 1966)

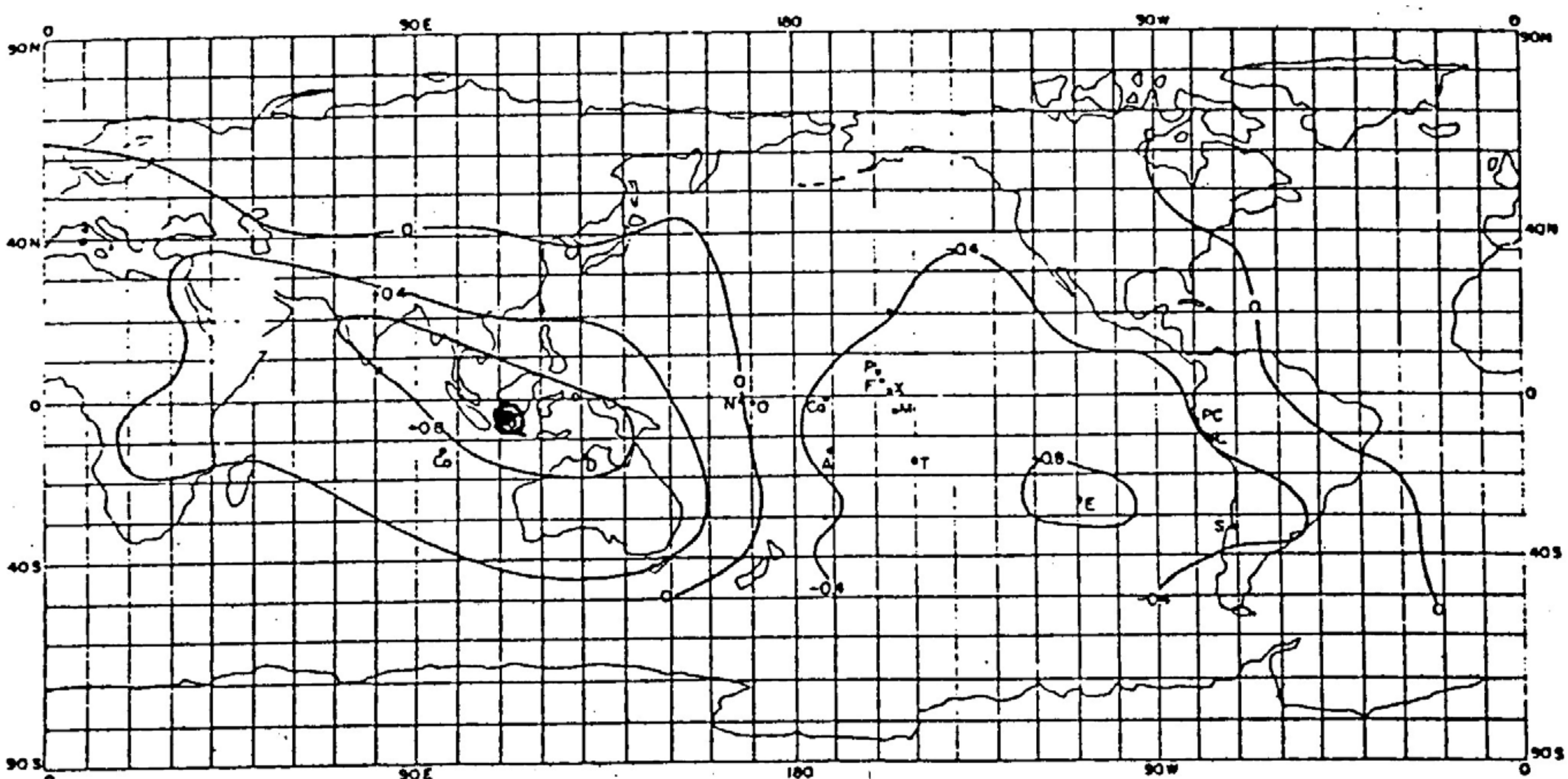


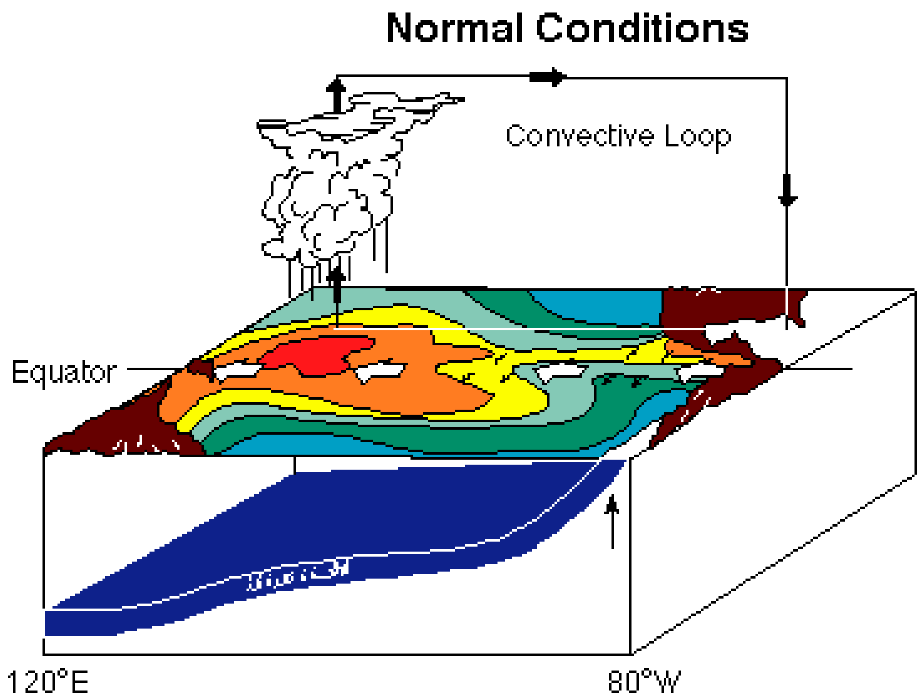
FIG. 1. Schematic map (after Berlage, 1966) showing isopleths of correlation of monthly mean station pressure with that of Djakarta, Indonesia (Dj). Other localities shown are Cocos Island (CO), Port Darwin (D), Nauru (N), Ocean Island (O), Palmyra (P), Christmas Island (X), Fanning (F), Malden Island (M), Apia, Samoa (A), Tahiti (T), Easter Island (E), Puerto Chicama (PC), Lima (L) and Santiago (S).

Southern Oscillation Index (SOI)

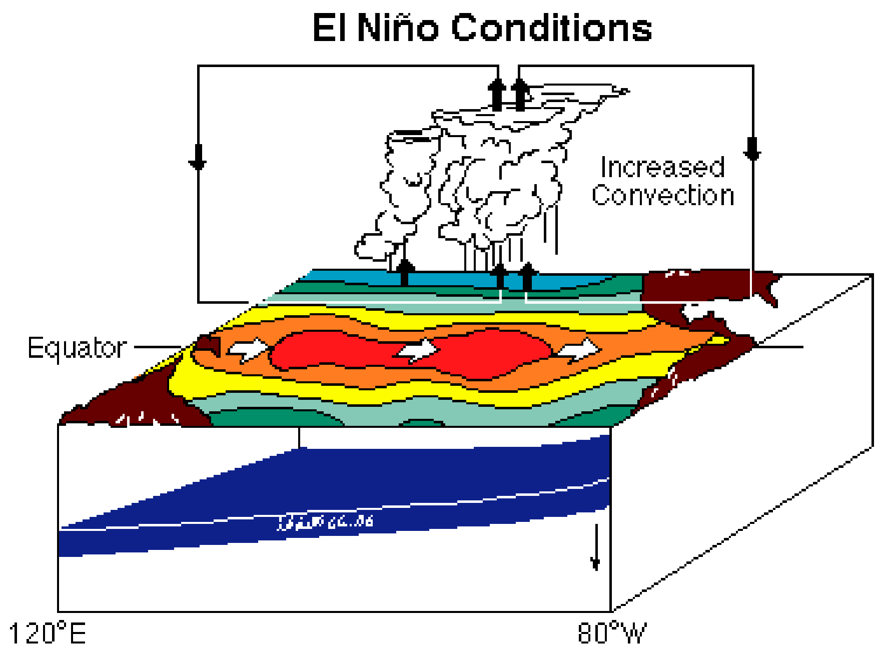
$$P_{\text{diff}} = p_{\text{Tahiti}} - p_{\text{Dawin}}$$

$$s(t) = 10 \times \frac{p_{\text{diff}}(t) - \langle p_{\text{diff}} \rangle}{SD(p_{\text{diff}})}$$

Positive SOI

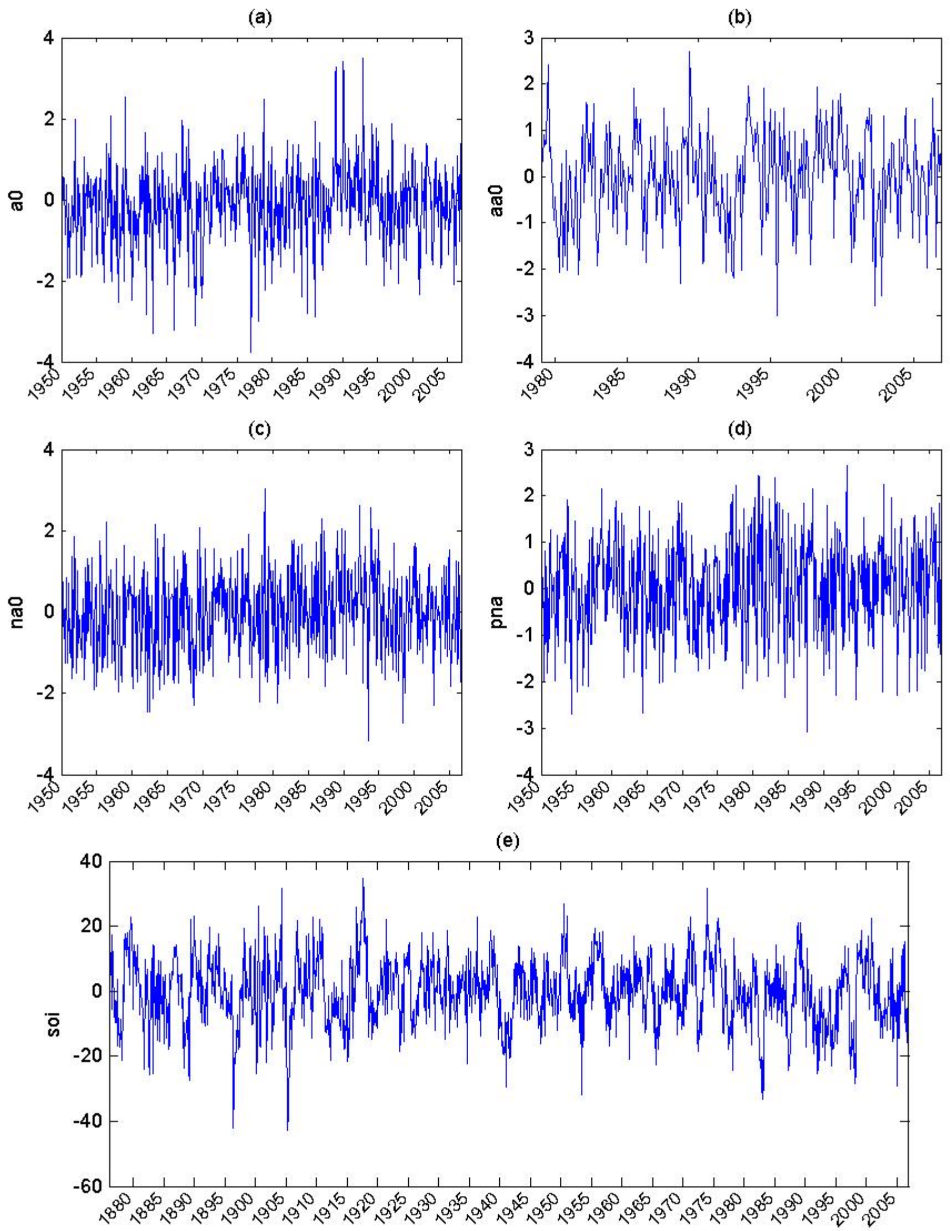


Sustained Negative SOI



Commonly Used Climate Indices

- Arctic Oscillation (AO)
- Antarctic Oscillation (AAO)
- North Atlantic Oscillation (NAO)
- Pacific/North American Pattern (PNA)
- Southern Oscillation Index (SOI)



Two Approaches → Index Prediction

- Forward Method
  - Collette and Ausloos (2004)
  - Lind et al. (2005)

- Backward Method
  - Chu (2007)

Forward Method

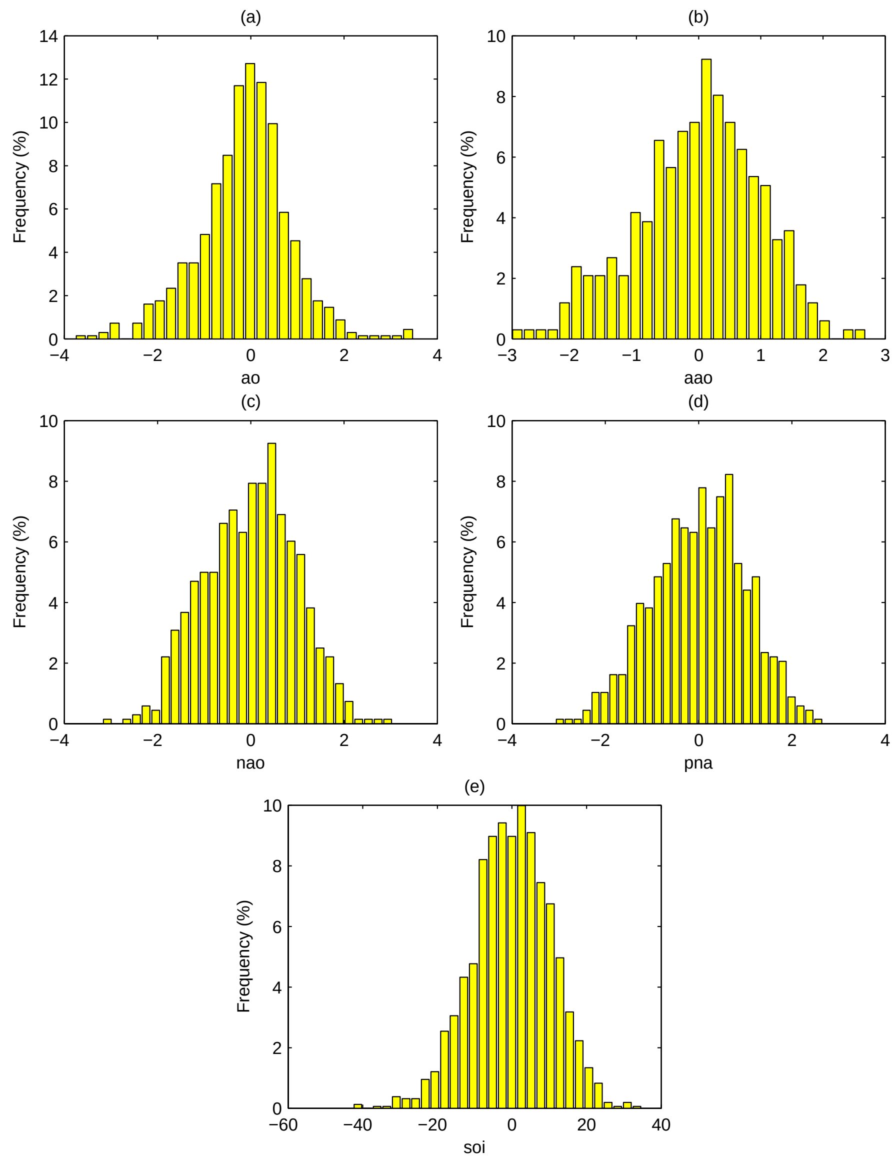
- Predicts the change of the index ρ at time *t* with a given temporal increment τ .
- Due to stochastic nature, the probability density function (PDF), should be first constructed.

PDFs of Monthly Mean Indices

- AO,   AAO

- NAO, PNA

- SOI



Example- NAO Index

- NAO describes a large-scale meridional vacillation in atmospheric mass between the anticyclone over Azores and the subpolar low pressure system over Iceland.

- Traditionally, the state of the NAO dipole system is characterized by an index, the so-called NAO index, which is basically the difference between the pressure at the high NAO pole (Azores) and the pressure at the low pole (Iceland).

Example – NAO Index $\rho(t)$

- Collette and Ausloos (2004) →
  - Brownian fluctuation

- Lind et al. (2005) → $\rho(t)$ → Langevin equation

$$\frac{d\rho(t)}{dt} = D^{(1)}\left[\rho(t),t\right] + \eta(t)\sqrt{D^{(2)}\left[\rho(t),t\right]}$$

($D^{(1)}$, $D^{(2)}$) → (Drift, Diffusion) Coefficients

$\eta(t)$ is a Langevin force
(δ - correlated Gaussian noise).

Backward Method

- This method predicts the typical time span (τ) needed to generate a fluctuation in the index of a given increment (ρ).

- This method uses FPT.

FPT Problem

- Given a fixed value of an index reduction (ρ), the corresponding time span (positive) is estimated for which the index reduction

$$\gamma_{\Delta t}(t) = s(t + \Delta t) - s(t)$$

reaches the level for the first time,

$$\tau_{\rho}(t) = \inf\{\Delta t > 0 \mid \gamma_{\Delta t}(t) \le -\rho\}$$

which is called the FPT. FPT is a random variable.

- PDF of FPT

$$p(\tau_\rho)$$

- CDF of FPT

$$P(\tau_\rho) = \int_{\tau_\rho}^{\infty} p(\tau)\,d\tau$$

Moments of FPT

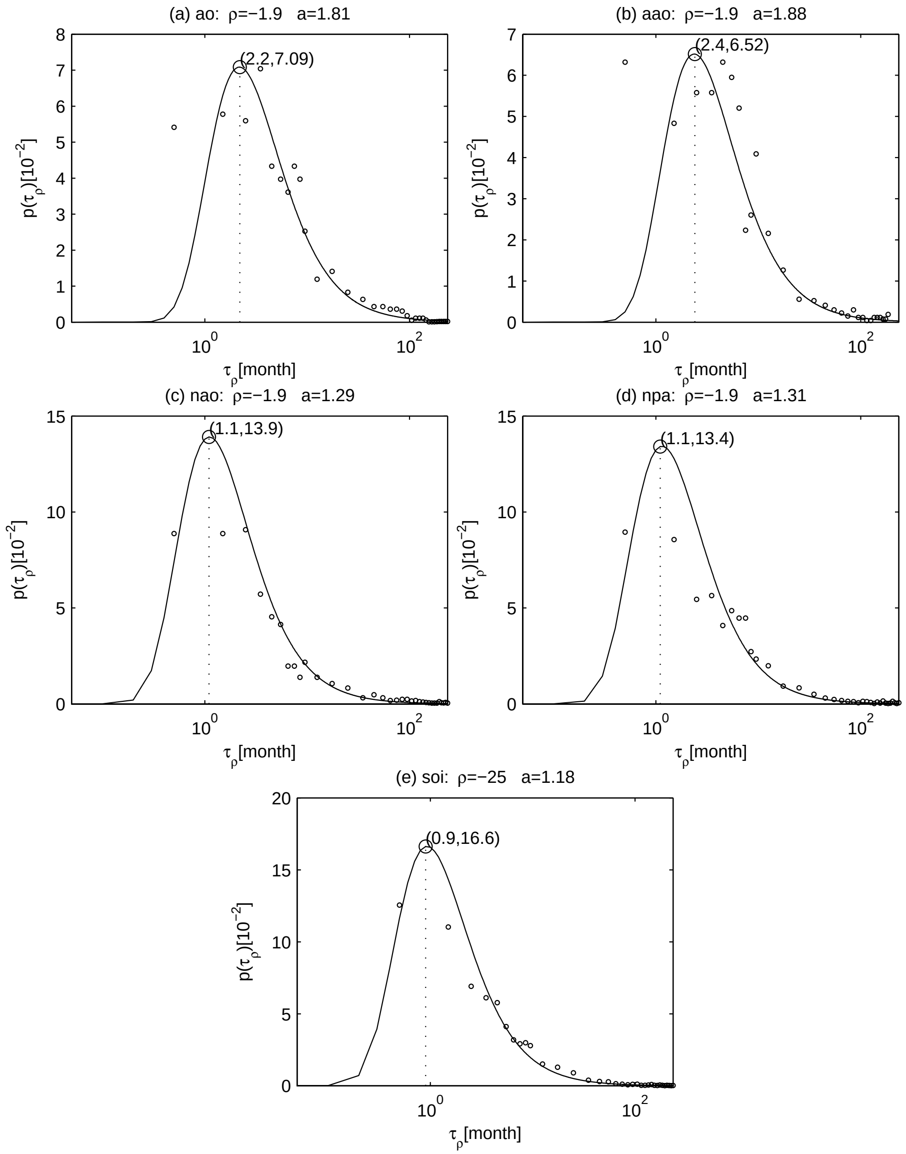
$$M_k(\rho) = k\int_0^\infty p(\tau)\tau^{k-1}d\tau, \qquad k = 1,\ldots,\infty$$

$$\langle \tau_\rho \rangle = M_1$$

$$\langle \delta\tau_\rho^2 \rangle = M_2 - M_1^2$$

PDF of FPT (from index data) → Inverse Gaussian Distribution

- AO AAO
- NAO PNA
- SOI



PDF of FPT
(Analytical, Chu et al. 2002a)

- Backward Fokker-Planck equation

$$\frac{\partial p}{\partial t} - \left[D^{(1)}(\rho,t) \right] \frac{\partial p}{\partial \rho} - \frac{1}{2} \eta^{(2)} D^{(2)}(\rho,t) \frac{\partial^2 p}{\partial \rho^2} = 0.$$

Analytical Solution of the Backward Fokker-Planck Equation

- For Brownian fluctuation (e.g., NAO monthly index, Collette and Ausloos 2004), the backward Kokker-Planck Equation has analytical solution (Ding and Rangaranjian 1995)

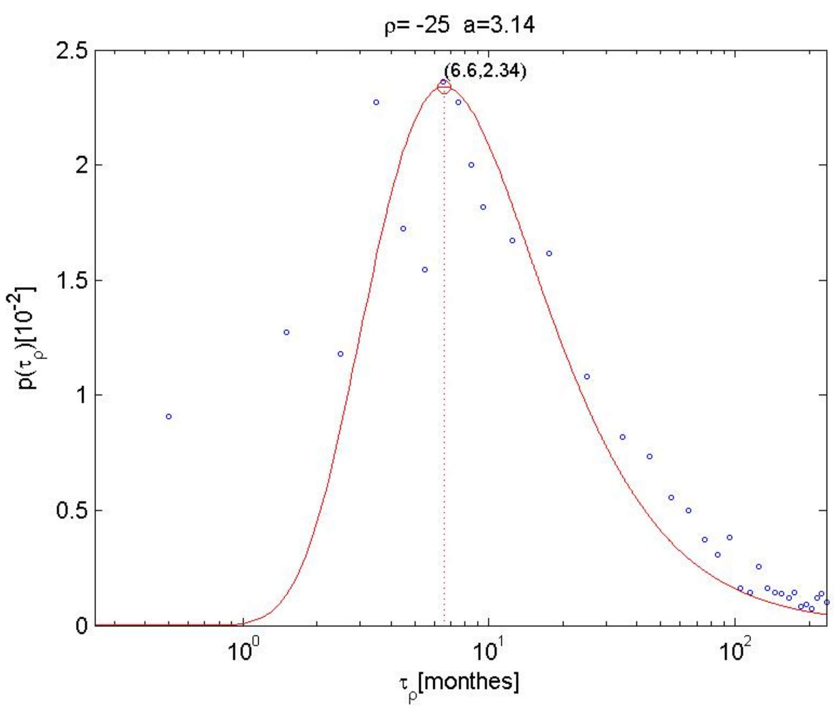
$$p(\tau) = \frac{1}{\sqrt{\pi}} \frac{a}{\tau^{3/2}} \exp\left(-\frac{a^2}{\tau} \right)$$

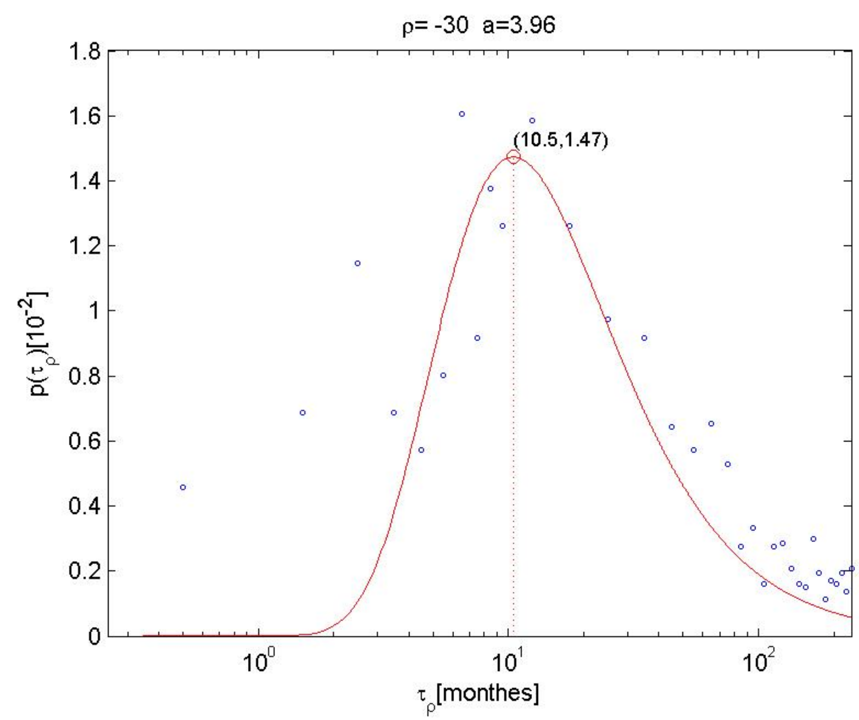
- Inverse Gaussian Distribution
- The parameter 'a' depends on the index reduction ρ

Dependence of $p(\tau_\rho)$ on ρ (SOI)

ρ = -25 → a = 3.14

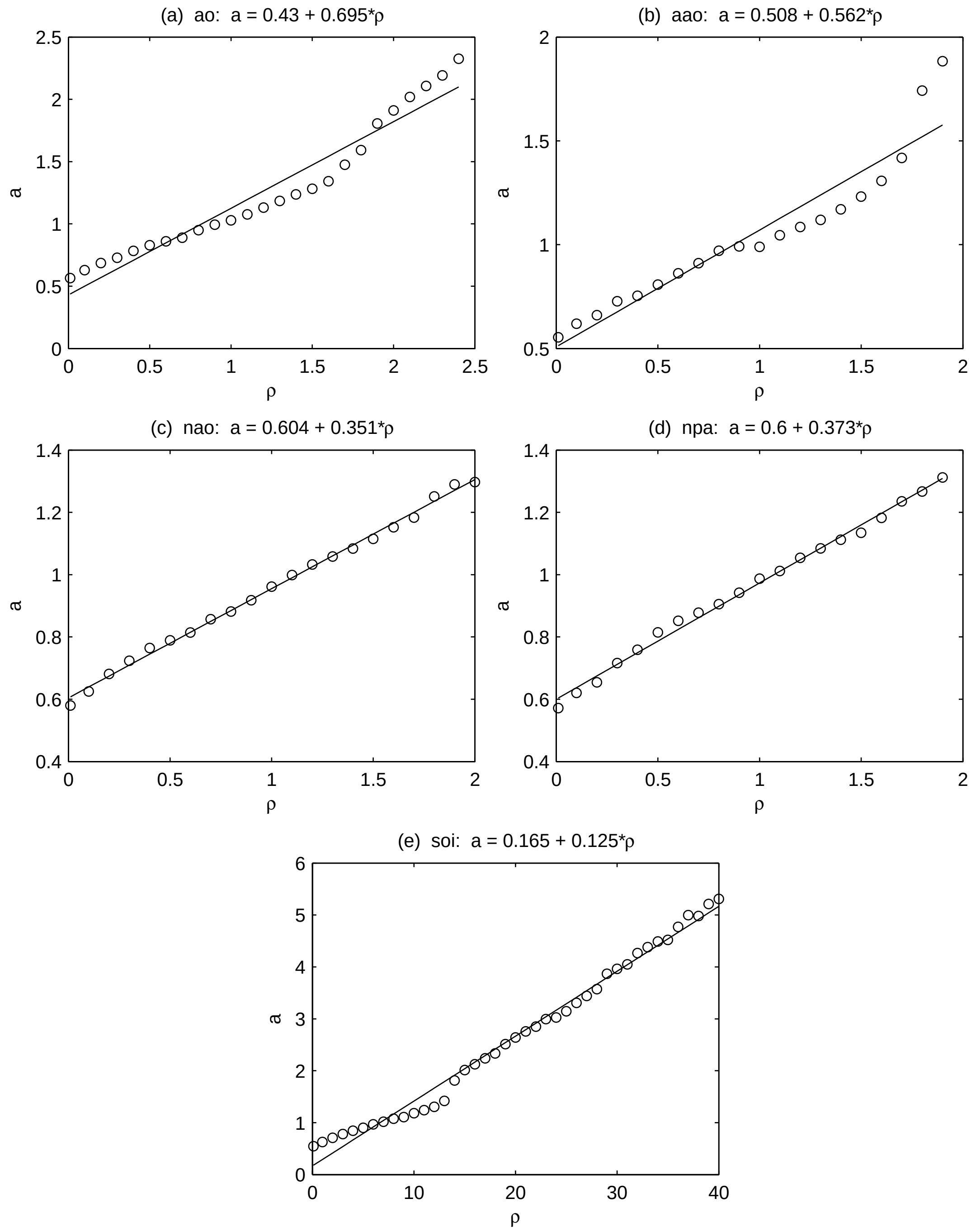
ρ = -30 → a = 3.96





- Linear relationship between the parameter 'a' in the analytical PDF & the index reduction ρ

$$a = \alpha_1 + \alpha_2 \rho$$



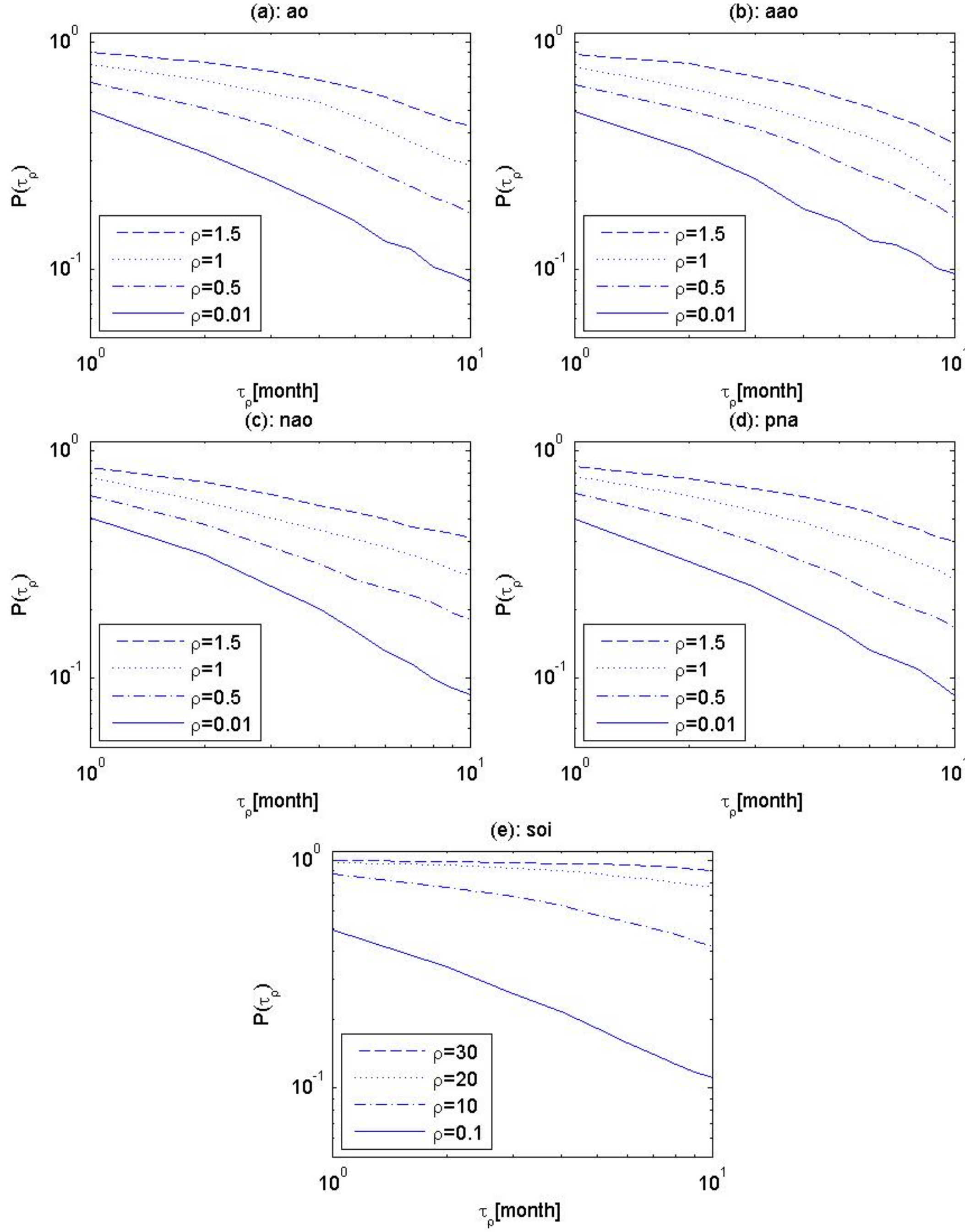
	AO	AAO	NAO	PNA	SO
α_1	0.430	0.508	0.604	0.600	0.165
α_2	0.696	0.562	0.351	0.373	0.125

- Empirical CDF of FPT $P(\tau_\rho)$

for various values of index reduction

→ Power Law

$$P(\tau) \sim \tau^{H-1}$$

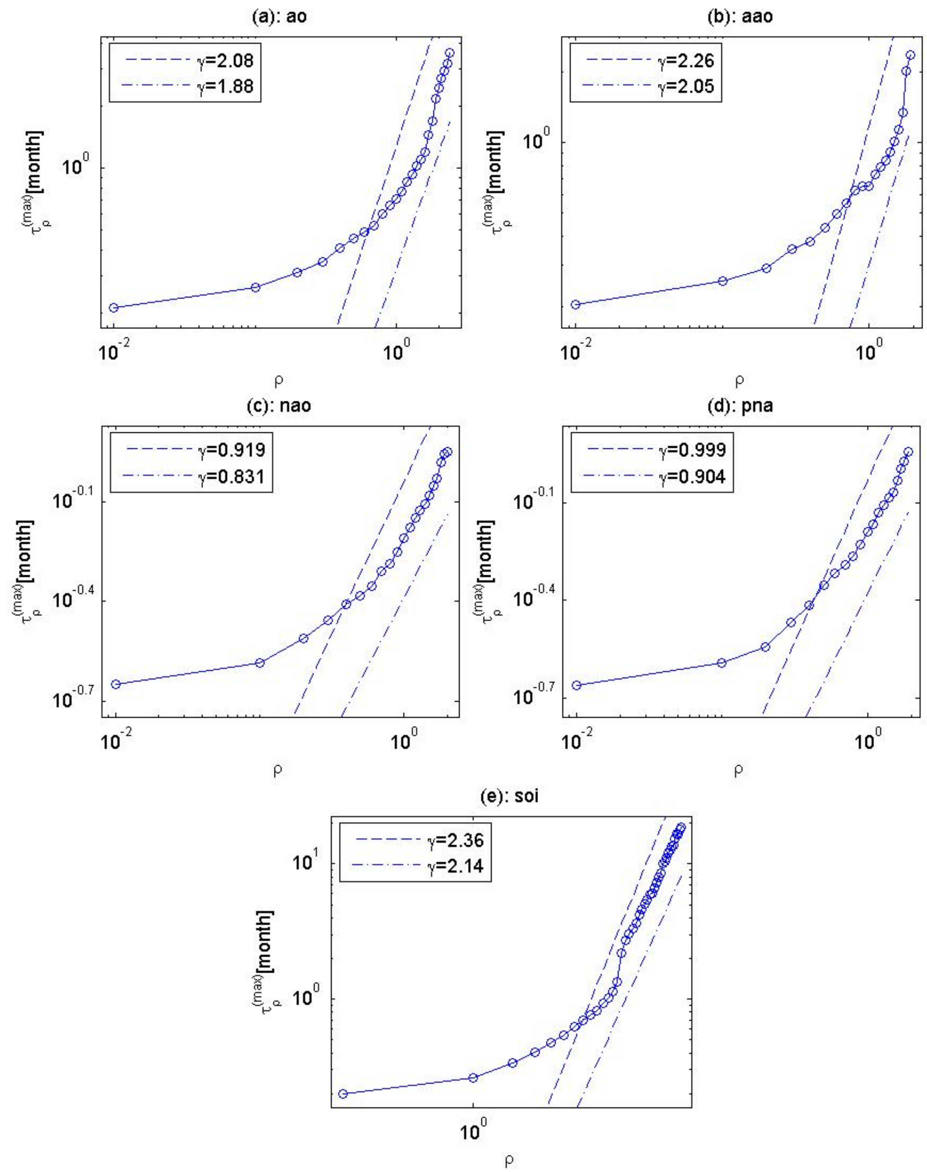


Mode of $p(\tau_\rho)$ → Optimal FPT $\tau_\rho^{(\max)}$

$\tau_\rho^{(\max)} = 2a^2/3$

$\tau_\rho^{(\max)} \sim \rho^\gamma$ for large ρ

$\gamma \sim 2.0$



Results

- FPT presents a new way to detect the temporal variability of the climate indices. It predicts a typical time span (τ) needed to generate an index reduction of a given increment (ρ).
- FPTs for the five climate indices satisfy the inverse Gaussian distribution → Brownian Fluctuation.
- $\tau_{\rho}^{(\max)}$ can be used as most probable time period needed for the low-frequency atmospheric circulation pattern to sustain.
- Power-law features

(2) Climate Model Predictability

This question should be answered before running any model

- How long is an ocean (or atmospheric) model valid once being integrated from its initial state?

Physical Reality

- $\boldsymbol{Y}$

- Physical Law: $d\boldsymbol{Y}/dt = \boldsymbol{h}(\boldsymbol{y}, t)$

- Initial Condition: $\boldsymbol{Y}(t_0) = \boldsymbol{Y}_0$

- $\boldsymbol{X}$ is the prediction of **Y**

- $d\boldsymbol{X}/dt = \boldsymbol{f}(\boldsymbol{X}, t) + q(t)\,\boldsymbol{X}$

- Initial Condition: $\boldsymbol{X}(t_0) = \boldsymbol{X}_0$

- Stochastic Forcing:
- $\langle q(t) \rangle = 0$
- $\langle q(t)q(t') \rangle = q^2\delta(t-t')$

Model Error

$$Z = X - Y$$

Initial: $Z_0 = X_0 - Y_0$

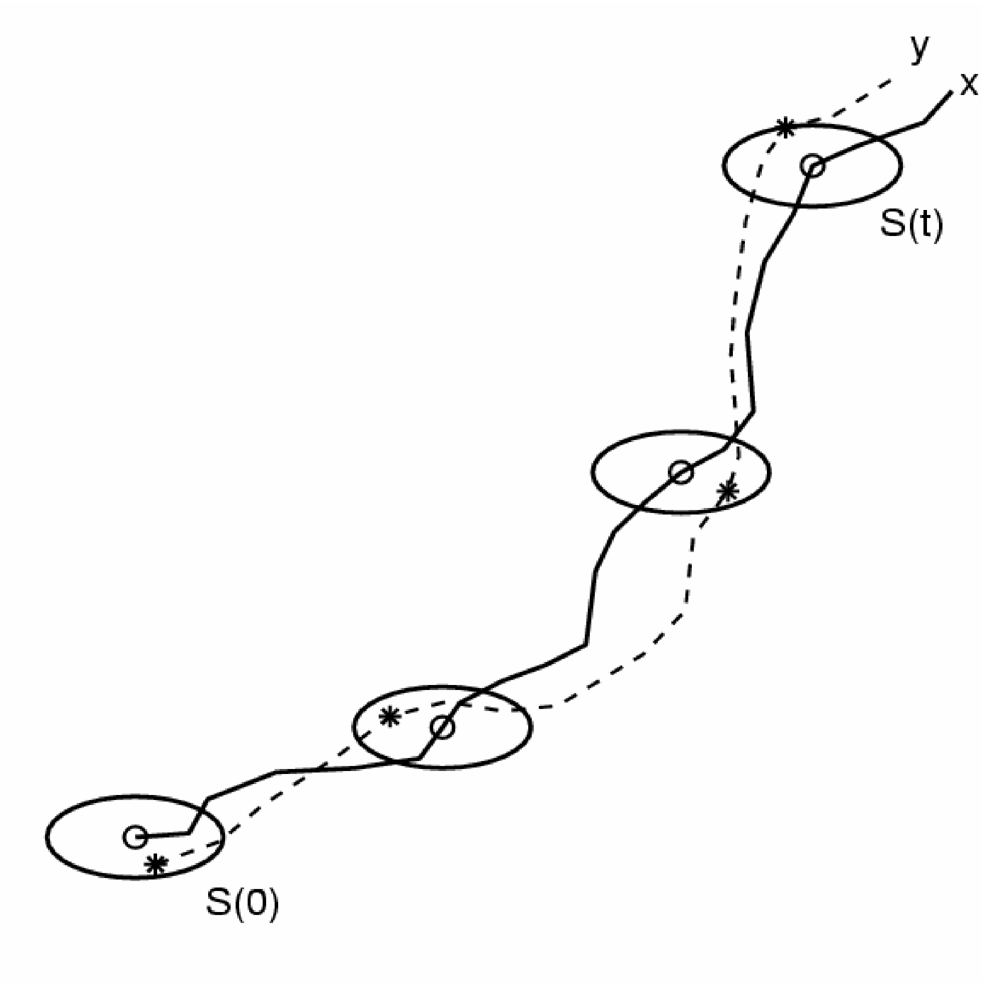
One Missing Parameter

- Tolerance Level ε
- Maximum accepted error

FPT

- FPT is defined as the time period when the prediction error first exceeds a pre-determined criterion (i.e., the tolerance level ε).

FPT for Model Predictability



FPT is defined as the time period when the prediction error first exceeds a pre-determined criterion (i.e., the tolerance level ε).

Conditional Probability Density Function

- Initial Error: $\boldsymbol{z}_0$
- $(t - t_0) \rightarrow$ Random Variable
- Conditional PDF of $(t - t_0)$ with given $\boldsymbol{z}_0 \rightarrow$

  $P[(t - t_0) \mid \boldsymbol{z}_0]$

Two Approaches

- Analytical (low dimension dynamical system)
- Practical (operational atmospheric or ocean model)

Backward Fokker-Planck Equation

$$\frac{\partial P}{\partial t} - \sum_{i=1}^{J}\left(f_i - \frac{d\hat{x}_i}{dt} \right)\frac{\partial P}{\partial z_i^0} - \frac{1}{2}\sum_{i=1}^{J}\sum_{j=1}^{J}\sum_{l=1}^{J} k_{il}k_{lj}\,\frac{\partial^2 P}{\partial z_i^0 \partial z_j^0} = 0,$$

Moments

$$\tau_1(\mathbf{z}_0) = \int_{t_0}^{\infty} P(t_0, \mathbf{z}_0, t - t_0)(t - t_0)\,dt$$

$$\tau_2(\mathbf{z}_0) = \int_{t_0}^{\infty} P(t_0, \mathbf{z}_0, t - t_0)(t - t_0)^2\,dt$$

Example-1: Maximum Growing Maniford of Lorenz System (Nicolis, 1992)

-

$$\frac{d\xi}{dt} = (\sigma - g\xi^2) + v(t)\xi\,, \qquad 0 \le \xi < \infty$$

$$\langle v(t) \rangle = 0, \qquad \langle v(\mathrm{t}) v(\mathrm{t}') \rangle = q^2 \delta(t - t')\,.$$

$$\sigma = 0.64, \qquad g = 0.3\,, \qquad q^2 = 0.2\,.$$

Mean and Variance of FPT

$$(\sigma\xi_0 - g\xi_0^2)\frac{d\tau_1}{d\xi_0} + \frac{q^2\xi_0^2}{2}\frac{d^2\tau_1}{d\xi_0^{\,2}} = -1$$

$$(\sigma\xi_0 - g\xi_0^2)\frac{d\tau_2}{d\xi_0} + \frac{q^2\xi_0^2}{2}\frac{d^2\tau_2}{d\xi_0^{\,2}} = -2\tau_1$$

$$\tau_1 = 0, \qquad \tau_2 = 0 \qquad\qquad \text{for} \quad \xi_0 = \varepsilon\,.$$

$$\frac{d\tau_1}{d\xi_0} = 0, \qquad \frac{d\tau_2}{d\xi_0} = 0 \qquad\qquad \text{for} \;\; \xi_0 = \xi_{\text{noise}}.$$

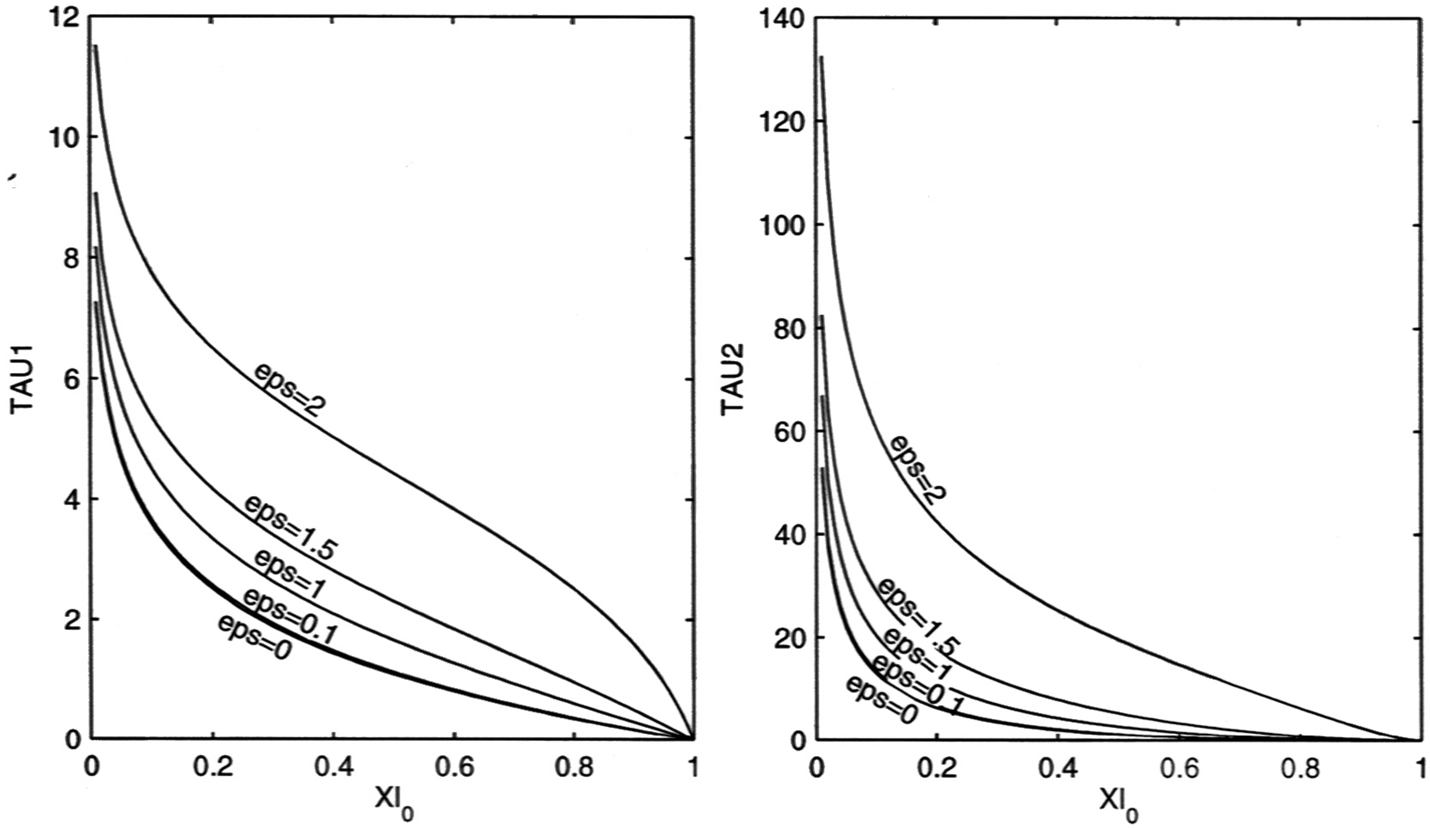
Analytical Solutions

$$\tau_1(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon) = \frac{2}{q^2} \int_{\bar{\xi}_0}^{1} y^{-\frac{2\sigma}{q^2}} \exp\left(\frac{2\varepsilon g}{q^2} y\right) \left[\int_{\bar{\xi}_{noise}}^{y} x^{\frac{2\sigma}{q^2}-2} \exp\left(-\frac{2\varepsilon g}{q^2} x\right) dx \right] dy$$

$$\tau_2(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon) = \frac{4}{q^2} \int_{\bar{\xi}_0}^{1} y^{-\frac{2\sigma}{q^2}} \exp\left(\frac{2\varepsilon g}{q^2} y\right) \left[\int_{\bar{\xi}_{noise}}^{y} \tau_1(x) x^{\frac{2\sigma}{q^2}-2} \exp\left(-\frac{2\varepsilon g}{q^2} x\right) dx \right] dy$$

$\bar{\xi}_0 = \xi_0 / \varepsilon$, $\bar{\xi}_{noise} = \xi_{noise} / \varepsilon$

Dependence of tau1 & tau2 on Initial Condition Error (ξ_0/ε)



Example-2: Lorenz System (1984) Modified Hadley Circulation

$$\frac{dx_1}{dt} = -x_2^2 - x_3^2 - ax_1 + aF,$$

$$\frac{dx_2}{dt} = x_1x_2 - bx_1x_3 - x_2 + G,$$

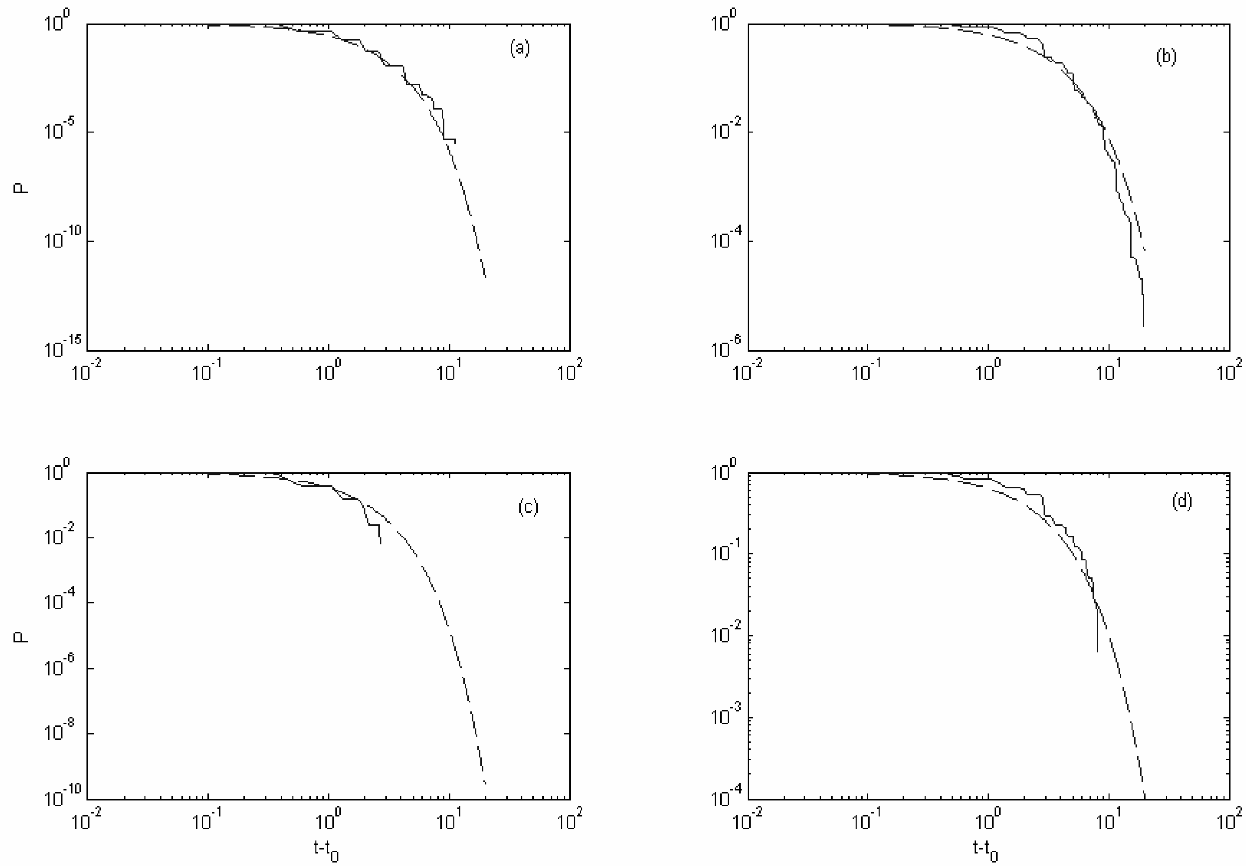
$$\frac{dx_3}{dt} = bx_1x_2 + x_1x_3 - x_3,$$

Sensitivity of PDF on the tolerance level and ensemble dimension: numerical (solid curve), analytic (dashed curve)

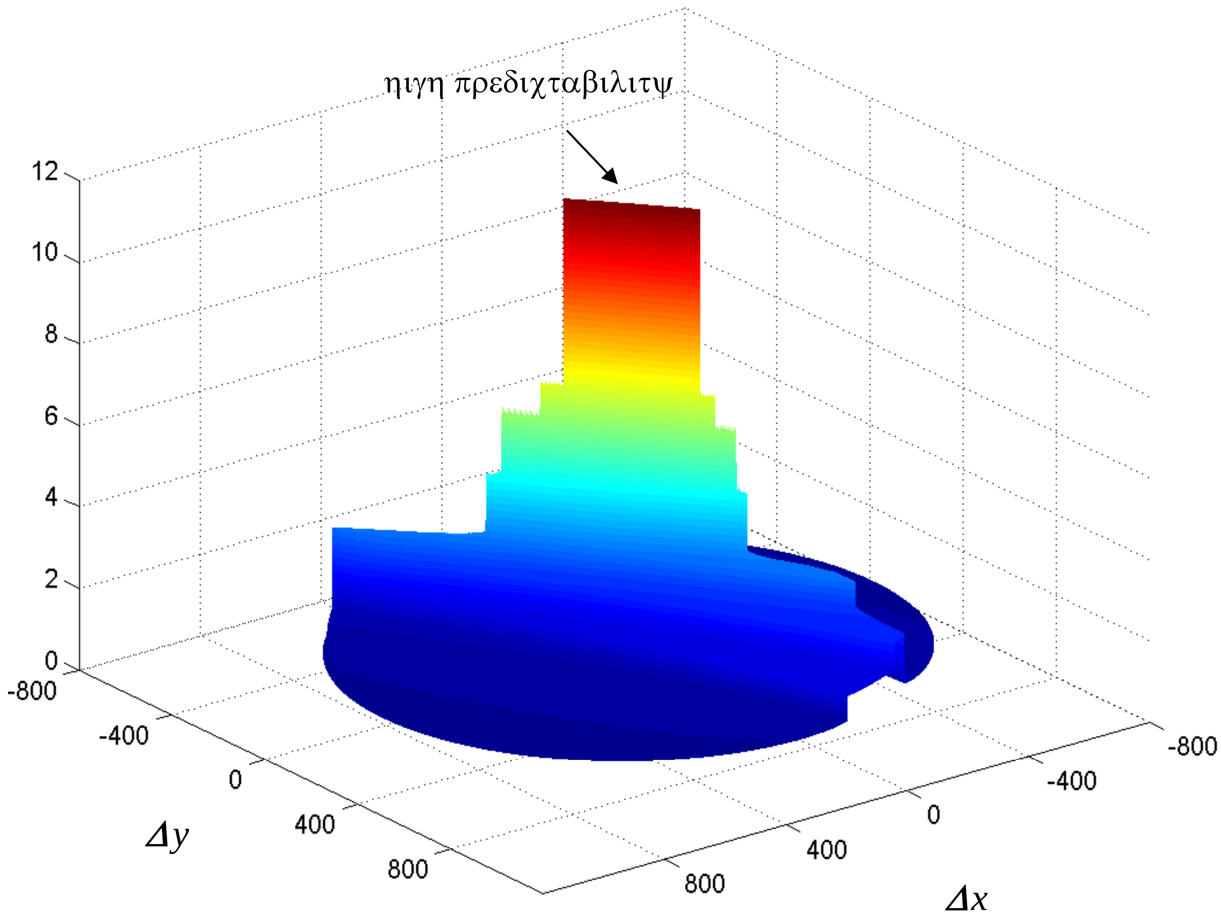
$\varepsilon=0.1$ $\varepsilon=0.5$

N=250,000

N=100



Dependence of FPT on Initial Error (Dx, Dy) for $\varepsilon = 0.1$



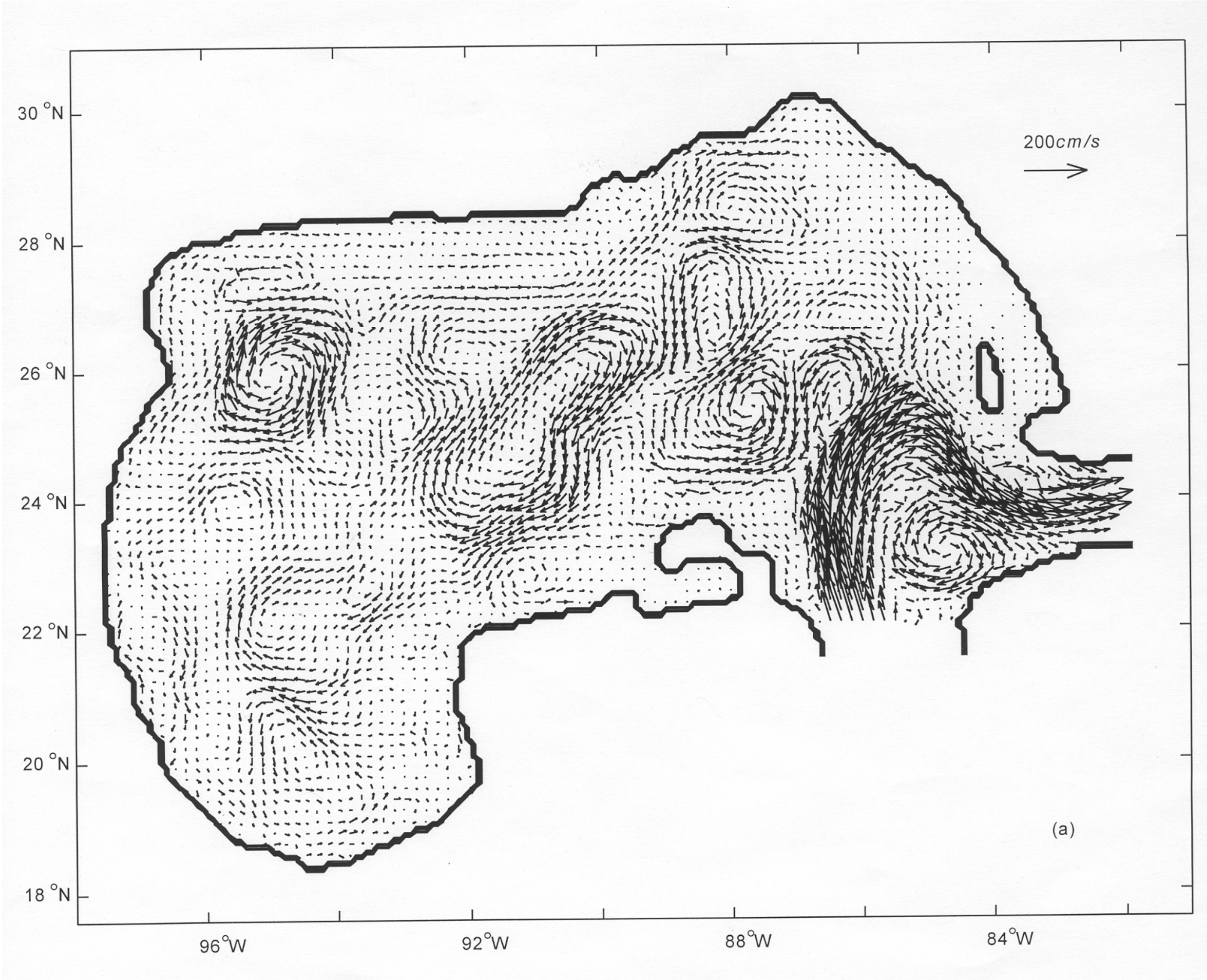
Example-3
Power Law Decay of Model Predictability Skill

Gulf of Mexico Prediction System

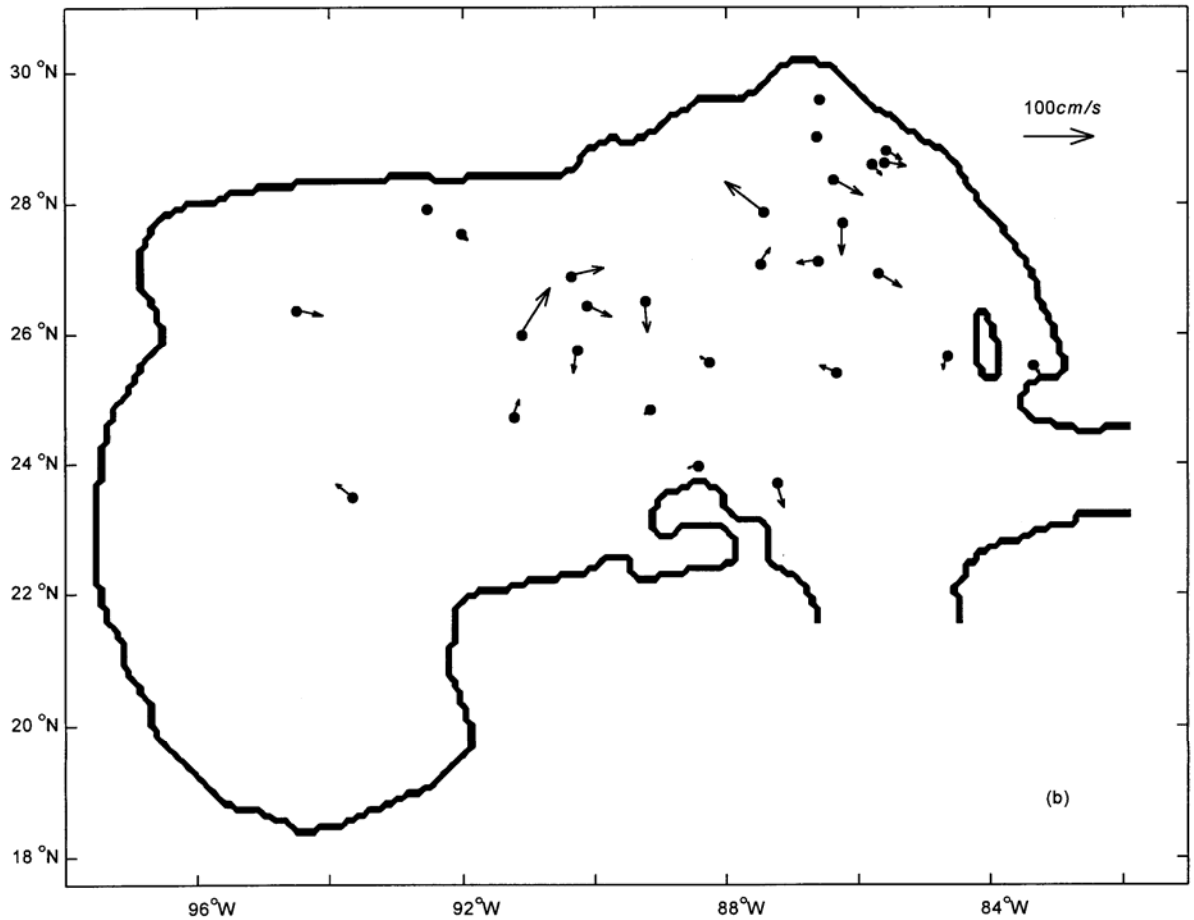
Gulf of Mexico Forecast System

- University of Colorado Version of POM
- 1/12^{o} Resolution
- Real-Time SSH Data (TOPEX, ESA ERS-1/2) Assimilated
- Real Time SST Data (MCSST, NOAA AVHRR) Assimilated
- Six Months Four-Times Daily Data From July 9, 1998 for Verification

Model Generated Velocity Vectors at 50 m on 00:00 July 9, 1998



(Observational) Drifter Data at 50 m on 00:00 July 9, 1998



Error Mean and Variance

Error Mean $$\mathbf{L}_1 = \langle \mathbf{z} \rangle$$

Error Variance $$\mathbf{L}_2 = \left\langle \left(\mathbf{z} - \langle \mathbf{z} \rangle \right)^t \left(\mathbf{z} - \langle \mathbf{z} \rangle \right) \right\rangle$$

Exponential Error Growth

$$L_1 \propto e^{\sigma t}, \qquad L_2 \propto e^{\omega t},$$

Classical Linear Theory

No Long-Term Predictability

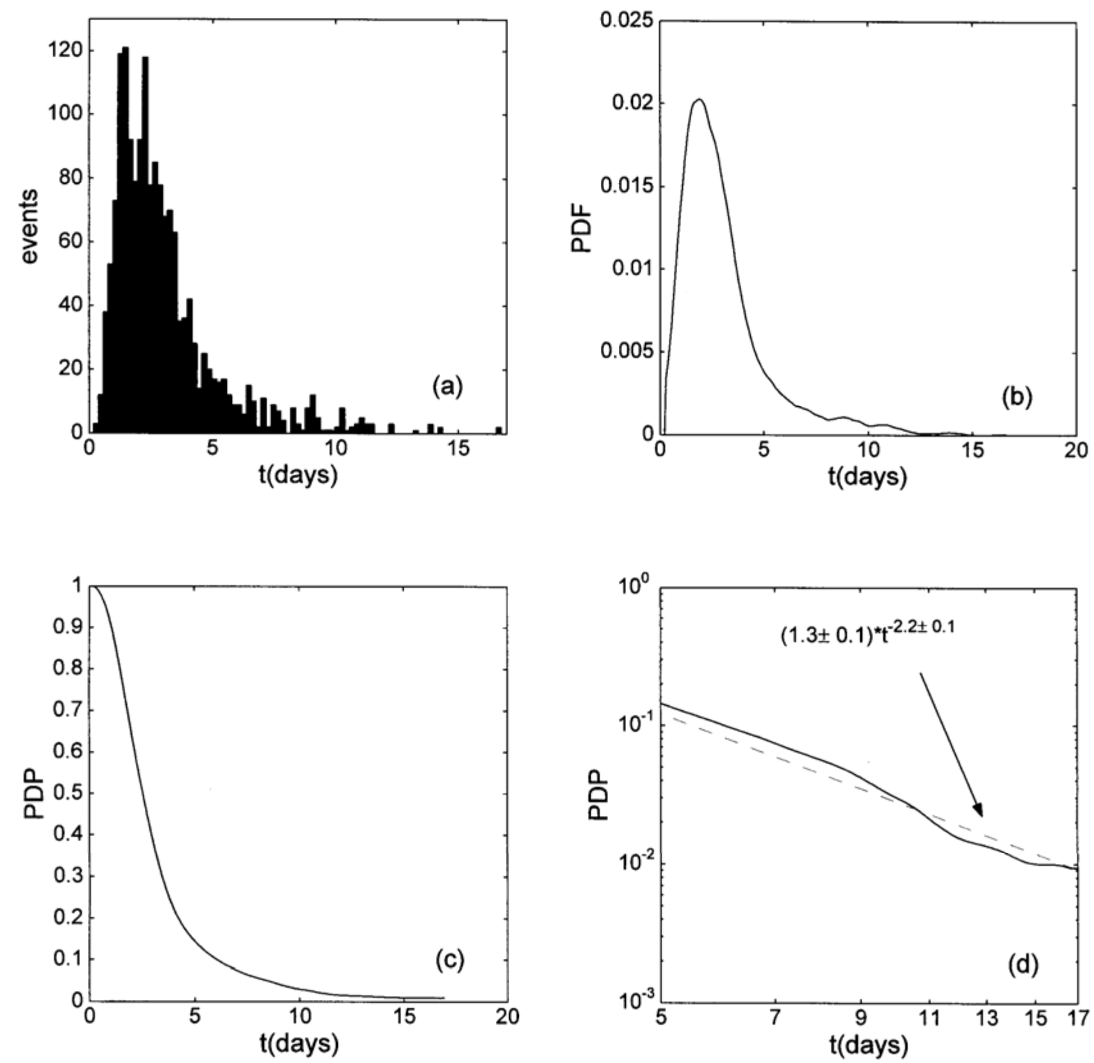
Power Law

$$L_1 \propto t^{\alpha}, \qquad L_2 \propto t^{\beta},$$

$$P(t_0, \mathbf{z}_0, \varepsilon, t - t_0) \sim t^{-\gamma} \quad \text{for large } t.$$

Long-Term Predictability May Occur

Statistical Characteristics of VPP for zero initial error and 22 km tolerance level (Non-Gaussion)

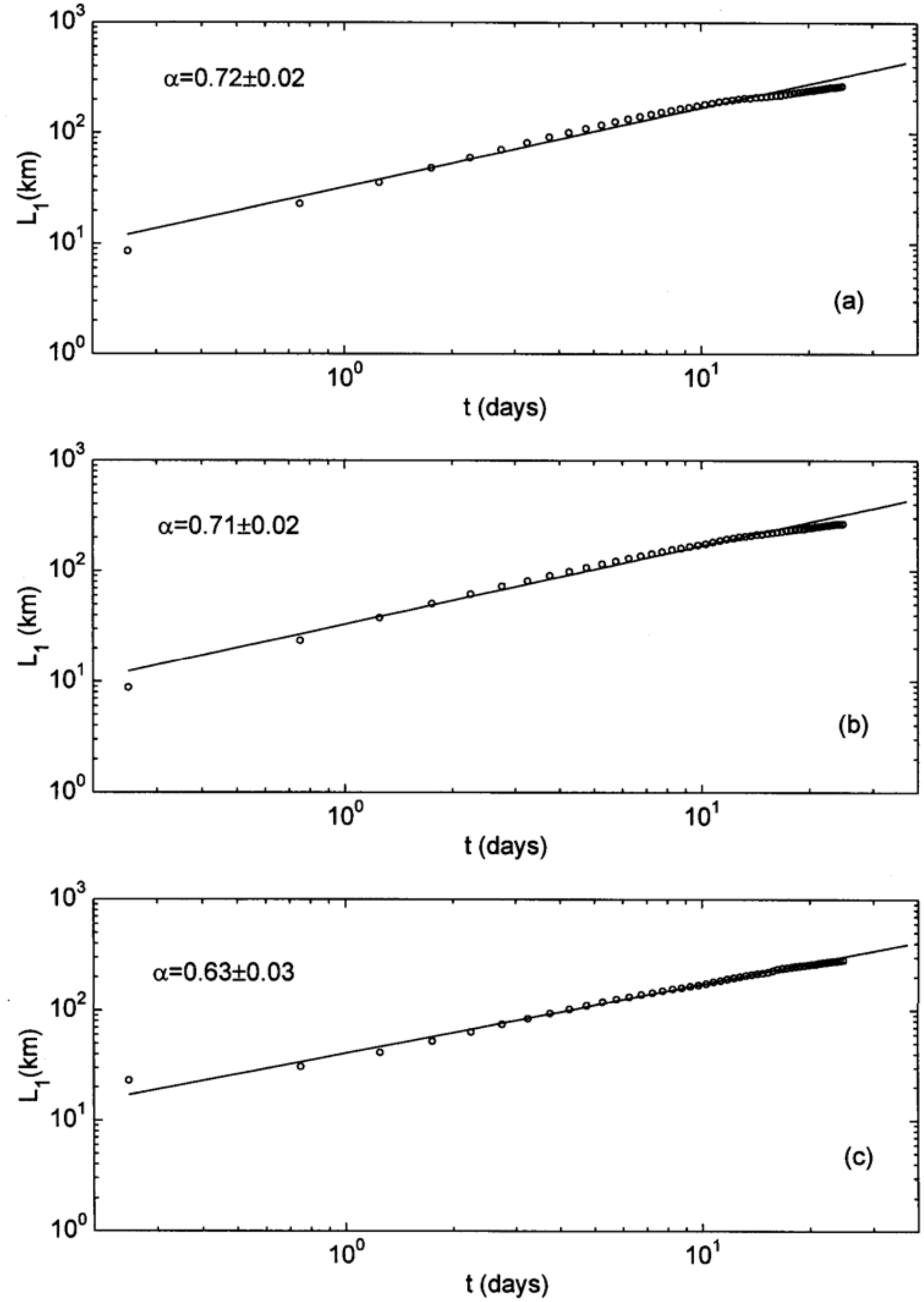


Scaling behavior of the mean error (L_1) growth for initial error levels:

(a) 0

(b) 2.2 km

(c) 22 km



Scaling behavior of the

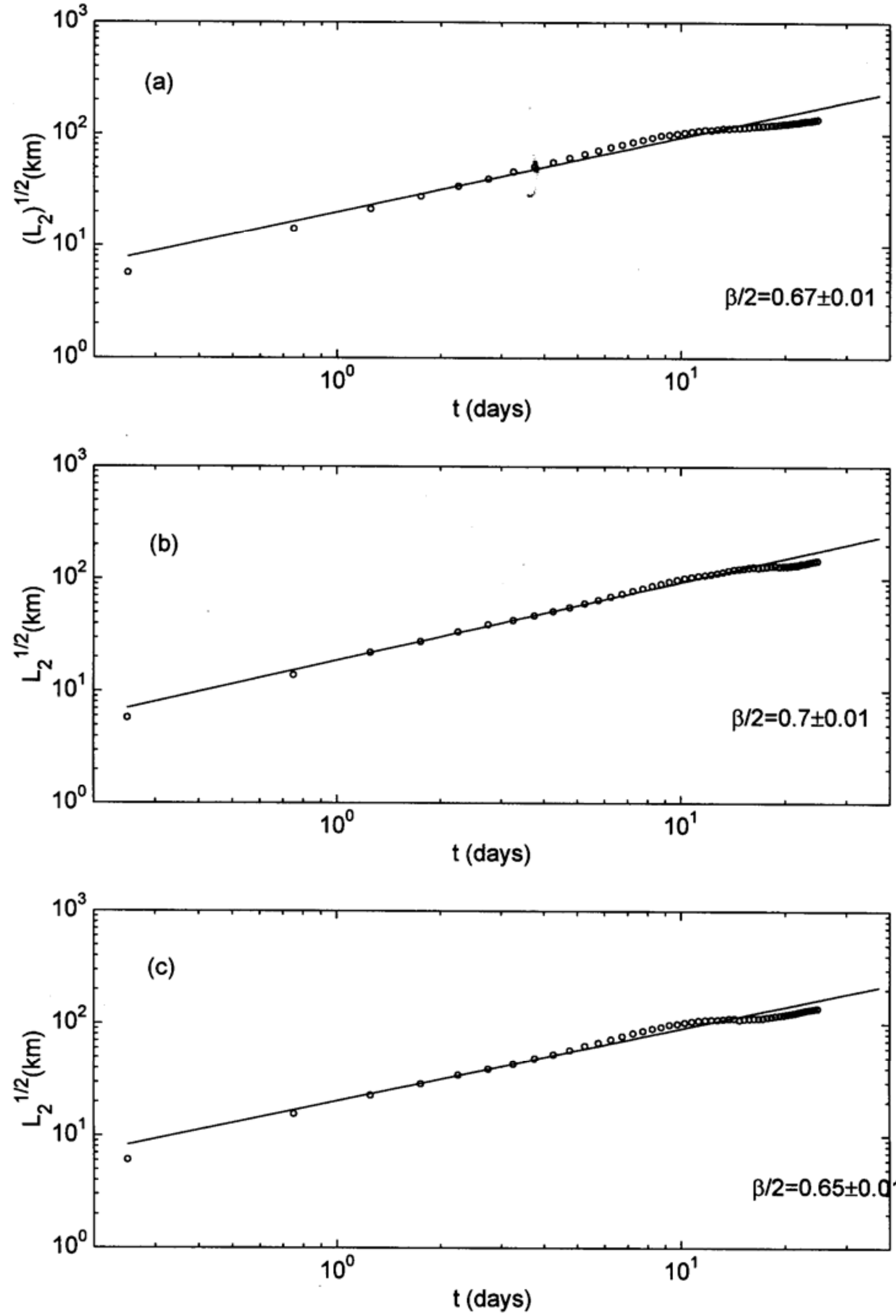
Error variance (L_2) growth

for initial error levels:

(a) 0

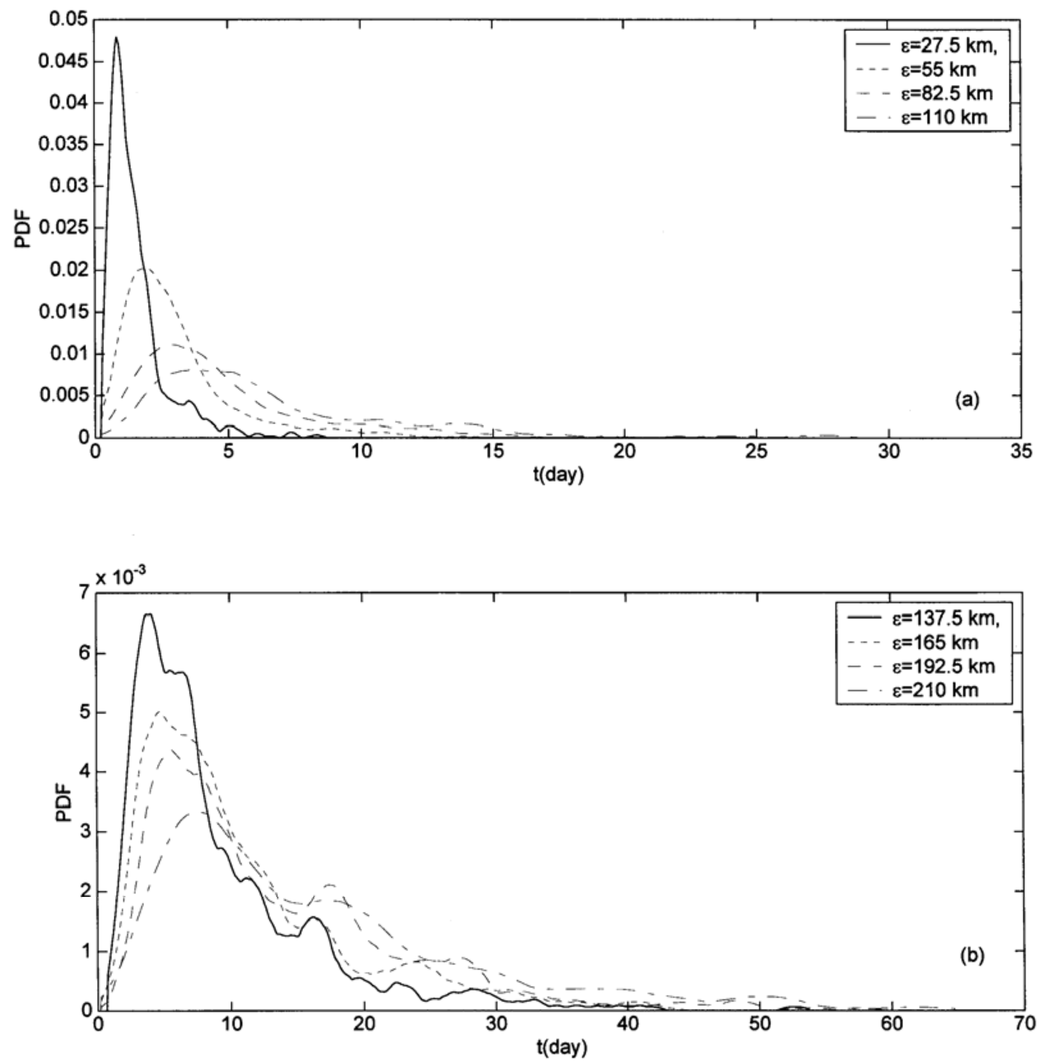
(b) 2.2 km

(c) 22 km



Probability Density Function of VPP calculated with different tolerance levels

Non-Gaussian distribution with long tail toward large values of VPP (Long-term Predictability)



The FPT analysis is usually conducted in the phase space.

Phase space representation

Results

- (1) FPT is an effective prediction skill measure (scalar).

- (2) Theoretical framework for FPT (such as Backward Fokker-Planck equation) can be directly used for model predictability study.

- (3) Spectral method is an effective way to transfer the data from the physical space into phase space (theoretical and practical significances).